Fourier Analysis

FOURIER SERIES

- Usually, a signal is described as a function of time .
- There are some amazing advantages if a signal can be expressed in the frequency domain.
- Fourier transform analysis is named after Jean Baptiste Joseph Fourier (1768-1830).

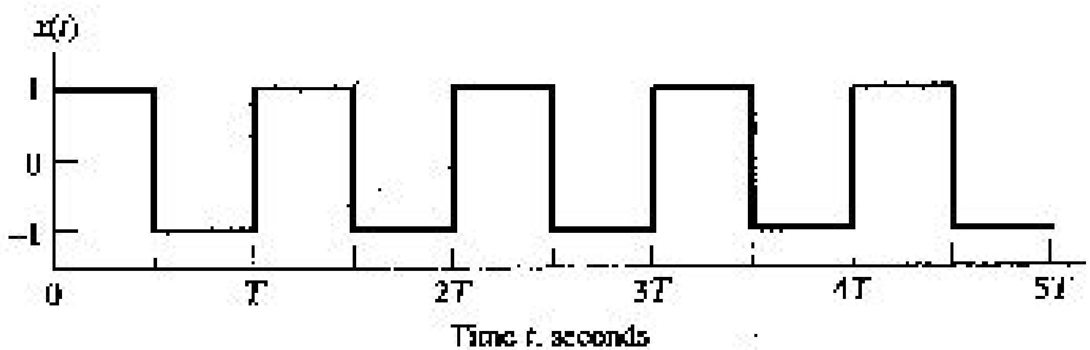
- A ***Fourier series*** (FS) is used for representing a continuous-time periodic signal as weighted superposition of sinusoids.
- **Periodic Signals** A continuous-time signal is said to be ***periodic*** if there exists a positive constant such that

$$x(t) = x(t + T_0)$$

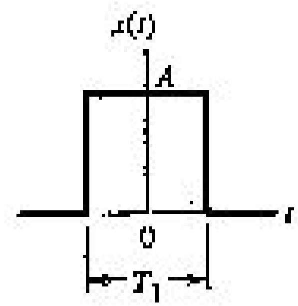
where T_0 is the period of the signal.

- T_0 : fundamental Period
- $f_0 = \frac{1}{T_0}$: fundamental frequency

- Example: Periodic and aperiodic signal

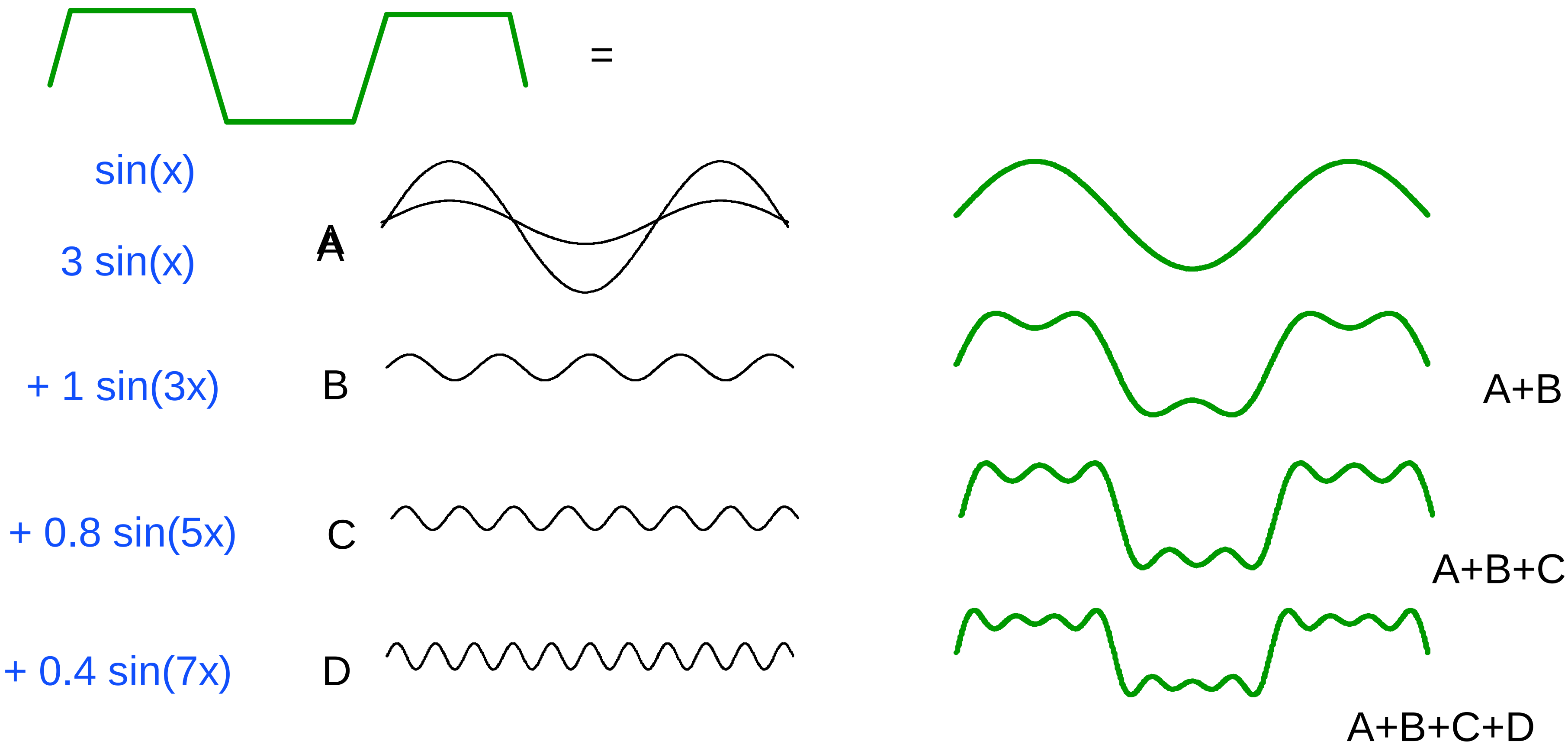


(a)



(b)

A sum of sines and cosines



Existence of the Fourier Series

- Existence

$$\int_0^{T_0} |f(t)|\,dt < \infty$$

- Convergence for all *t*

$$|f(t)| < \infty \quad \forall t$$

- Finite number of maxima and minima in one period of *f*(*t*)
- These are known as the Dirichlet conditions

Fourier Series

- General representation of a periodic signal

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- Fourier series coefficients

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)dt$$

- Polar Form of Fourier series

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

where $c_0 = a_0, c_n = \sqrt{a_n^2 + b_n^2}$, and

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

- *{xn}* are called the Fourier series coefficients of the signal *x(t).*
- The quantity $f_0 = \frac{1}{T_0}$ is called the fundamental frequency of the signal *x(t)*
- The Fourier series expansion can be expressed in terms of angular frequency $\omega_0 = 2\pi f_0$ by

$$x_n = \frac{\omega_0}{2\pi}\int_{\alpha}^{\alpha+2\pi/\omega_0} x(t)e^{-jn\omega_0 t}dt$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

- Discrete spectrum - We may write $x_n = |x_n| e^{j\angle x_n}$, where $|x_n|$ gives the magnitude of the nth harmonic and $\angle x_n$ gives its phase.

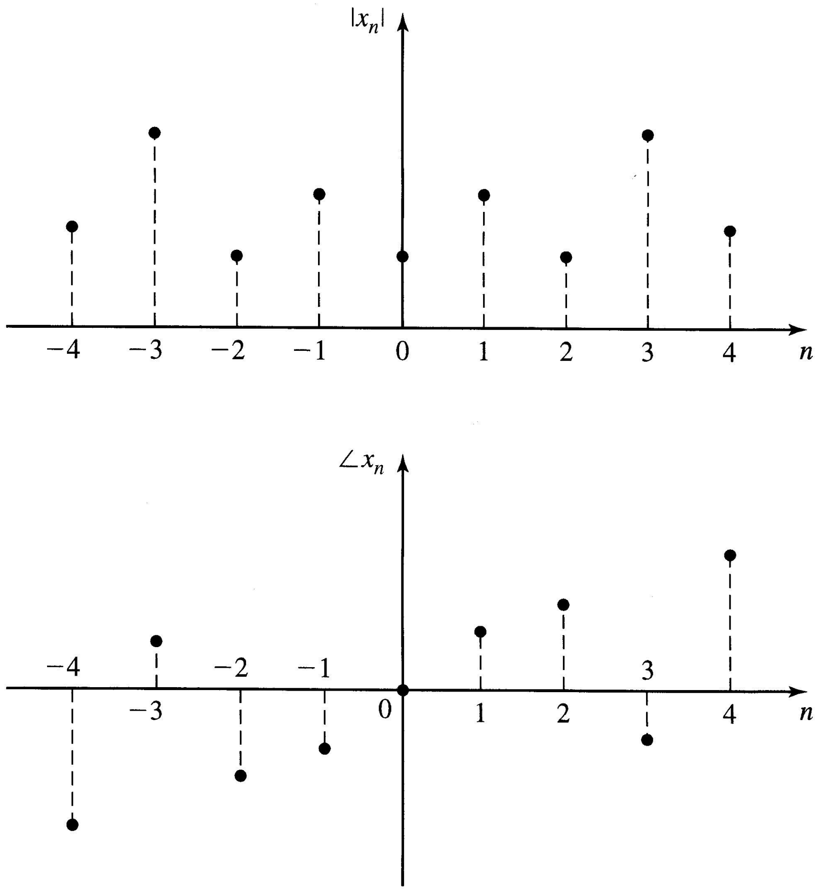


Figure 2.1 The discrete spectrum of $x(t)$.

- Example: Let *x(t)* denote the periodic signal depicted in Figure 2.2

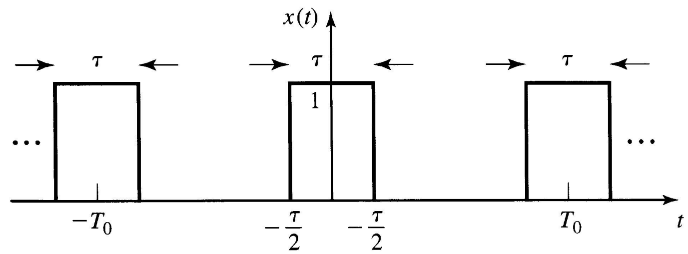


Figure 2.2 Periodic signal $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_0}{\tau}\right), \quad T_0 > \tau,$$

where

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

is a rectangular pulse. Determine the Fourier series expansion for this signal.

Solution: We first observe that the period of the signal is T_0 and

$$x_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1 e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\frac{T_0}{-jn2\pi}\left[e^{-jn\frac{n\tau}{T_0}} - e^{jn\frac{n\tau}{T_0}}\right]$$

$$= \frac{1}{\pi n}\sin\left(\frac{n\pi\tau}{T_0}\right)$$

$$= \frac{\tau}{T_0}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Therefore, we have

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}$$

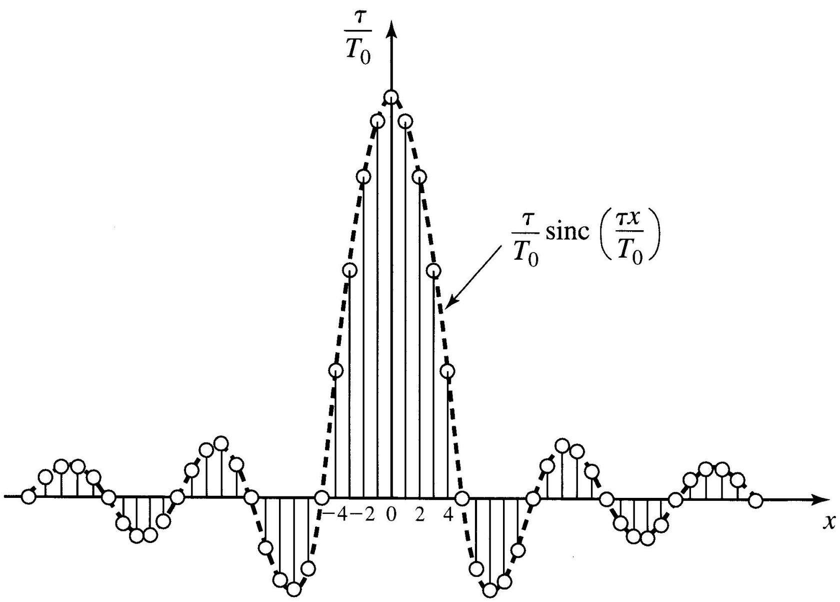
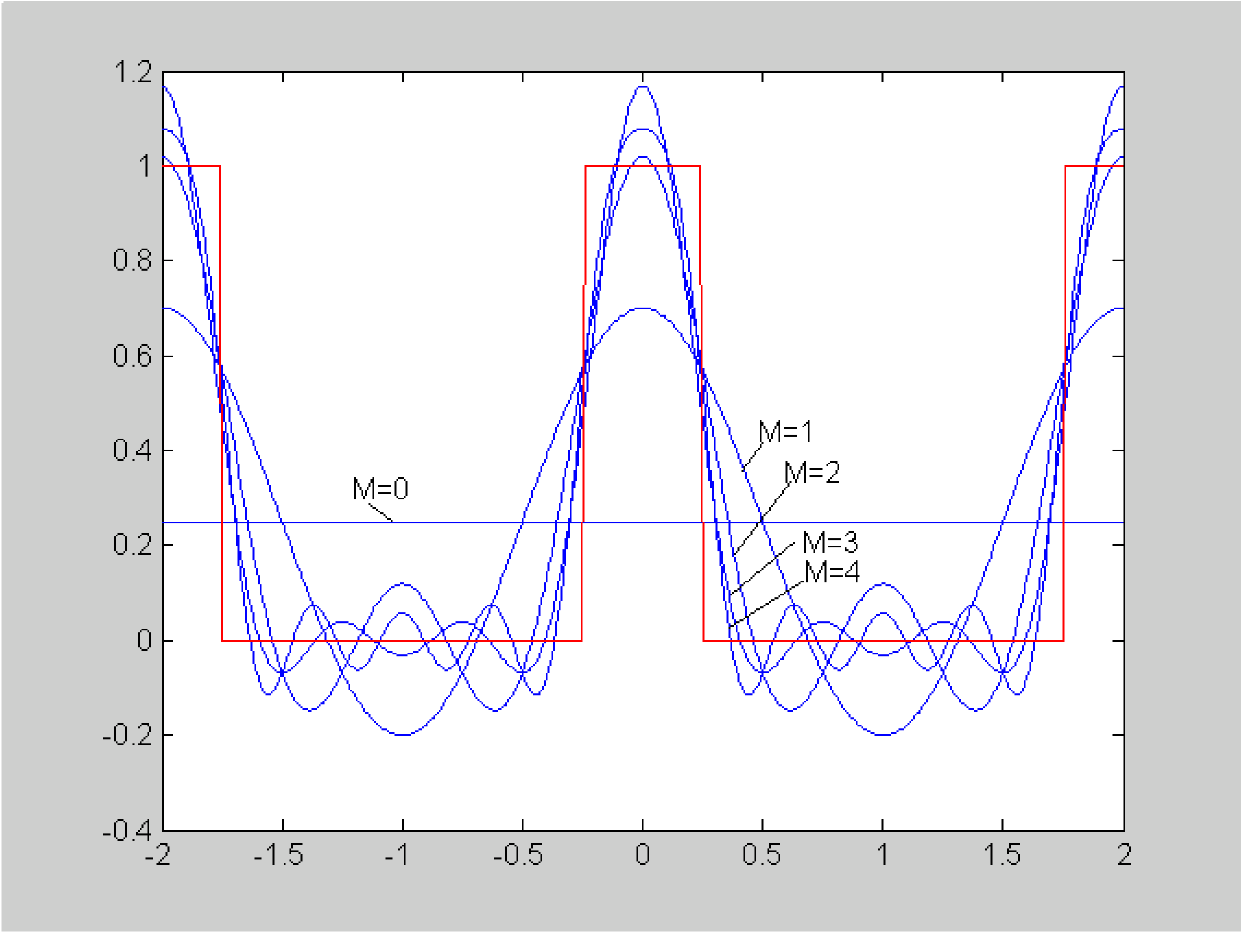
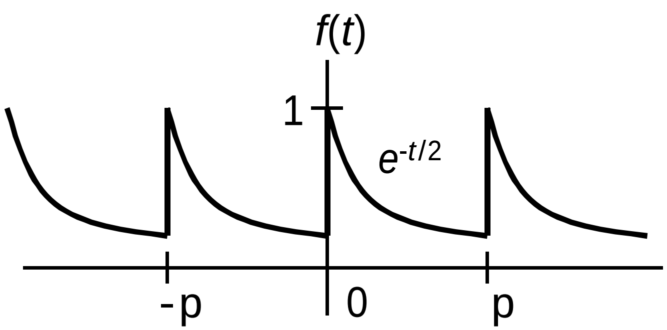


Figure 2.3 The discrete spectrum of the rectangular pulse train.



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
M=0
M=1
M=2
M=3
M=4

Example #1



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}dt = -\frac{2}{\pi}\left(e^{-\frac{\pi}{2}} - 1\right) \approx 0.504$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}\cos(2nt)\,dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}\sin(2nt)\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

a_n and b_n decrease in amplitude as $n \to \infty$.

- Fundamental period

  T_0 = p

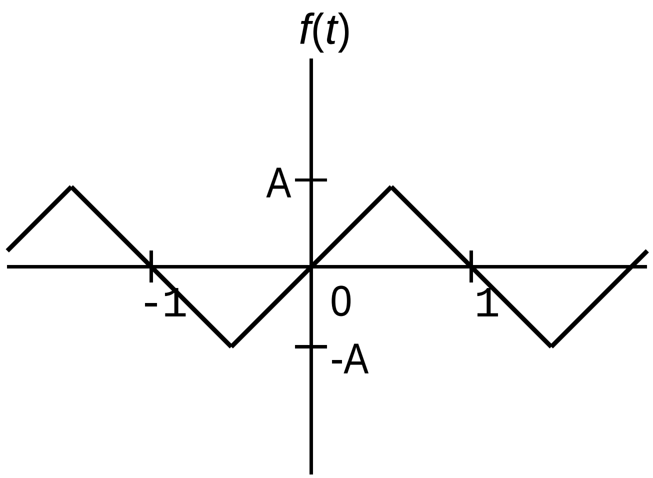
- Fundamental frequency

  f_0 = 1/T_0 = 1/p Hz

  w_0 = 2p/T_0 = 2 rad/s

$$f(t) = 0.504\left[1 + \sum_{n=1}^{\infty}\frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right]$$

Example #2



- Fundamental period

  $T_0 = 2$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/2$ Hz

  $w_0 = 2p/T_0 = p$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi\, n\, t) + b_n \sin(\pi\, n\, t)$$

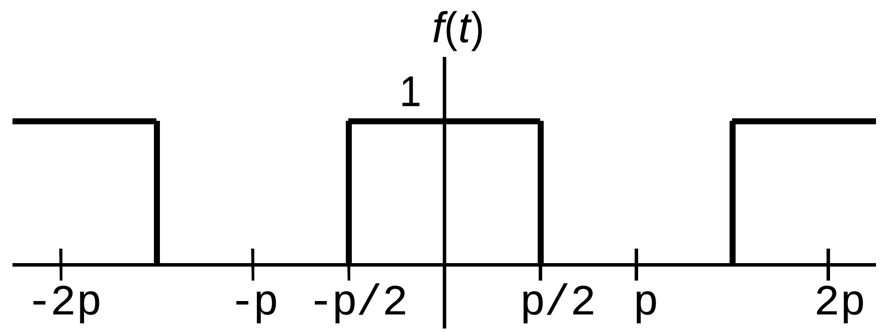
$$a_0 = 0 \quad \text{(by inspection of the plot)}$$

$$a_n = 0 \quad \text{(because it is odd symmetric)}$$

$$b_n = \frac{2}{\pi} \int_{-1/2}^{1/2} 2A\,t \sin(\pi\, n\, t)\, dt + \frac{2}{\pi} \int_{1/2}^{3/2} (2A - 2A\,t) \sin(\pi\, n\, t)\, dt$$

$$b_n = \begin{cases} 0 & n \text{ is even} \\ \dfrac{8A}{n^2\pi^2} & n = 1,5,9,13,\ldots \\ -\dfrac{8A}{n^2\pi^2} & n = 3,7,11,15,\ldots \end{cases}$$

Example #3



- Fundamental period

  T_0 = 2p

- Fundamental frequency

  f_0 = 1/T_0 = 1/2p Hz

  w_0 = 2p/T_0 = 1 rad/s

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3,7,11,15,\ldots \\ -\pi & n = 3,7,11,15,\ldots \end{cases}$$

Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Property	**Periodic Signal**	**Fourier Series Coefficients**
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$

Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$